Meeting 4 Math 22

Reading Debrief

- Discuss Preview Activity 9.4.1 with your group.
- Are there any questions your group wants to address?

Questions?

- Right hand rule?

Section 9.4.1 Computing the Cross Product

Definition 9.4.4 The cross product of vectors $u = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is the vector

$$u \times v = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}.$$

Determinants

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

You can use determinants to rewrite the formula for the cross product in a compact form:

$$u \times v = (u_2v_3 - u_3v_2)\hat{i} - (u_1v_3 - u_3v_1)\hat{j} + (u_1v_2 - u_2v_1)\hat{k}.$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Easier to remember

Properties of the Cross Product Let u, v, w be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$. Then

1. $u \times v = -(v \times u)$
2. $(u+v) \times w = (u \times w) + (v \times w)$
3. $(cu) \times v = c(u \times v) = u \times (cv)$
4. $u \times v = 0$ if and only if u and v are parallel.
5. The cross product is not associative.

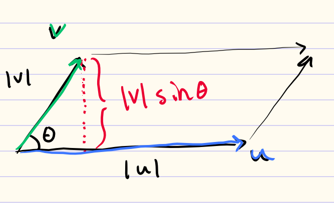
Section 9.4.2 The Length of $u \times v$

The Length of $u \times v$ Let u and v be vectors in $\mathbb{R}^3$. Then

$$|u \times v| = |u||v|\sin\theta$$

where $0 \le \theta \le \pi$ is the angle between u and v.

The formula looks alot like $u \cdot v = |u||v|\cos\theta$.



From the picture:

$$|u \times v| = |u||v|\sin\theta$$
= Area of parallelegram spanned by u and v

In particular, we see that

$|u \times v| = 0 \iff$ Area of parallegram is zero $\iff$ u and v are parallel

Activity 9.4.3

- Complete Activity 9.4.3 and discuss w/ your group.
- Class discussion.

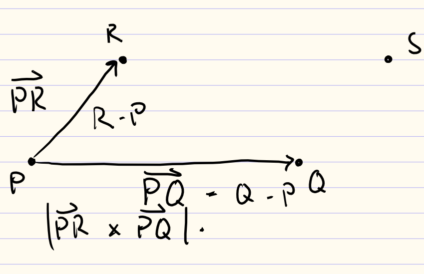
a. Compute $|u \times v|$.

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \hat{i}\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= 3\hat{i} - 7\hat{j} - 9k$$

$$\Rightarrow |u \times v| = \sqrt{3^2 + 7^2 + 9^2}$$

b. Strategy:



Then compute $|\vec{PR} \times \vec{PQ}|$.

Section 9.4.3 The Direction of $u \times v$

First, we want to check that $u \times v$ is perpendicular to both u and v.

We have

$$u \cdot (u \times v) = \langle u_1, u_2, u_3 \rangle \cdot \begin{vmatrix} \hat{i} & \hat{j} & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= u_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

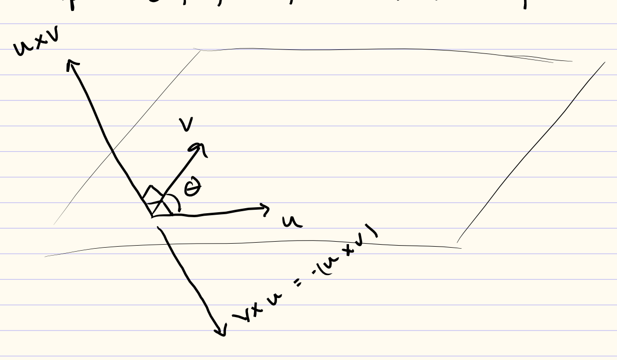
$$= \cancel{u_1u_2v_3} - \cancel{u_1v_2u_3} - \cancel{u_1u_2v_3} + \cancel{v_1u_2u_3} + \cancel{u_1v_2u_3} - \cancel{v_1u_2u_3}$$

$$= 0$$

So u and $u \times v$ are perpendicular. Then

$$v \cdot (u \times v) = v \cdot (-(v \times u))$$
$$= (-1)(v \cdot (v \times u))$$
$$= (-1) \cdot 0 = 0.$$

So v is perpendicular to $u \times v$. You can also show that the triple $(u, v, u \times v)$ satisfies right hand rule.

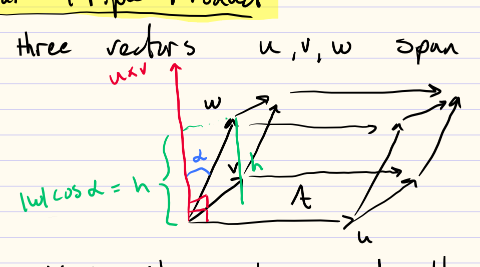


Geometric Definition of $u \times v$ Let u and v be vectors in $\mathbb{R}^3$. Then $u \times v$ is the vector in $\mathbb{R}^3$ such that

1. $u \times v$ is perpendicular to both u and v and $(u, v, u \times v)$ satisfies the right hand rule. (direction)
2. $|u \times v| = |u||v|\sin\theta$ (magnitude)
   = area of parallelogram spanned by u & v
   where $0 \le \theta \le \pi$ is the angle between u & v.

Scalar Triple Product

Any three vectors u, v, w span a parallelepiped.



Let V be the volume of the parallelepiped. Then

$$V = Ah = |u \times v||w|\cos\alpha$$
$$= |\underbrace{w \cdot (u \times v)}_{\text{Scalar triple product}}|$$

The volume of the parallelepiped spanned by u, v, w is equal to the absolute value of the scalar triple product $|(u \times v) \cdot w| = |w \cdot (u \times v)|$.

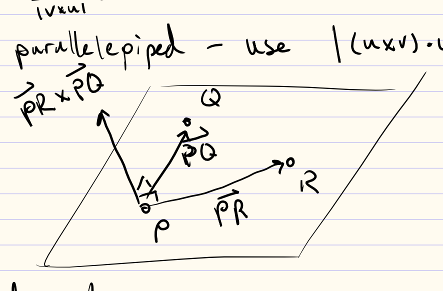
Activity 9.4.4

- Complete Activity 9.4.4 and discuss w/ your group.
- Class discussion.

a. $\frac{u \times v}{|u \times v|}$ and $\frac{v \times u}{|v \times u|}$.

b. Volume of parallelepiped - use $|(u \times v) \cdot w|$

c.



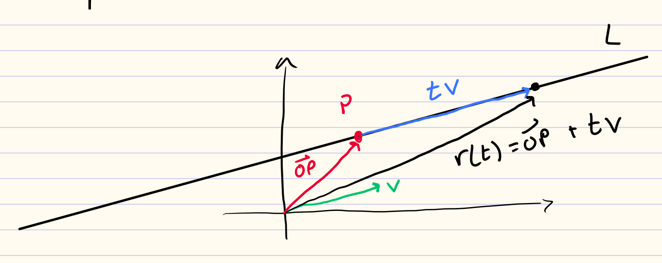
d. Right hand rule

e. u, v, w lie in the same plane if and only if the parallelepiped has zero volume.

End of Section 9.4.

Section 9.5.1 Lines and Planes

Definition 9.5.3 A line through a point P in the direction of a vector v is the set of terminal points of all vectors parallel to v with initial point P.



Vector form of a Line The vector form of the line through P in the direction of v is the vector-valued function

$$r(t) = \vec{OP} + tv.$$

Section 9.5.2 Parametric Equations of a Line

Write $\vec{OP} = \langle x_0, y_0, z_0 \rangle$ and $v = \langle a, b, c \rangle$. Then

$$r(t) = \langle x_0, y_0, z_0 \rangle + t\langle a, b, c \rangle$$
$$= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

Parametric Eq of a Line The functions

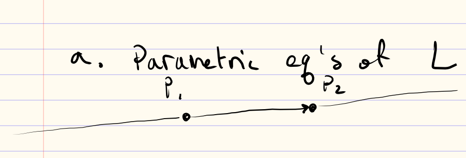
$$x(t) = x_0 + at \qquad y(t) = y_0 + bt \qquad z(t) = z_0 + ct$$

are called the parametric eq's of a line.

Activity 9.5.3

- Complete 9.5.3 and discuss w/ your group
- Class discussion.

a. Parametric eq's of L: $x(t) = 1 - 3t$, $y(t) = 2 - t$, $z(t) = -1 - t$ $\qquad v = \langle -3, -1, -1 \rangle$



b. Does $(1,2,1)$ lie on L? $(1,2,1)$ lies on L if there exists $t \in \mathbb{R}$ such that $1 = 1 - 3t$, $2 = 2 - t$, $1 = -1 - t$ $\Rightarrow$ no solution

c. Direction of K: $x(s) = 11 + 4s$, $y(s) = 1 - 3s$, $z(s) = 3 + 2s$. The direction is $v = \langle 4, -3, 2 \rangle$. The lines L and K are parallel if their direction vectors are parallel. Check if the cross product is zero.

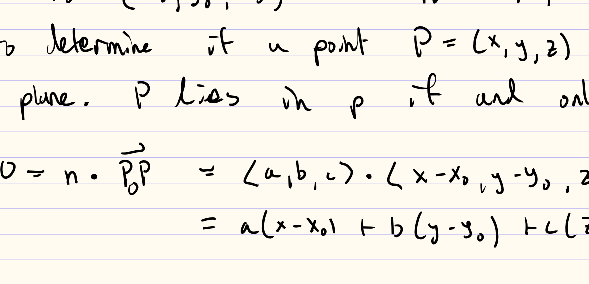
d. Do K and L intersect? K and L intersect if there exist $s, t \in \mathbb{R}$ such that

$$\left.\begin{aligned} 1 - 3t &= 11 + 4s \\ 2 - t &= 1 - 3s \\ -1 - t &= 3 + 2s \end{aligned}\right\} \text{ setting } x, y, z \text{ components equal.}$$

The solution is $(s, t) = (-1, -2)$. So the lines intersect.

Section 9.5.3 Planes in Space

Definition 9.5.6 A plane p through a point P_0 with normal vector n is the set of terminal points of all vectors with initial point P_0 that are perpendicular to n.



Suppose $P_0 = (x_0, y_0, z_0)$ and $n = \langle a, b, c \rangle$. We want to determine if a point $P = (x, y, z)$ lies in the plane. P lies in p if and only if

$$0 = n \cdot \vec{P_0P} = \langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0).$$

Equations of a Plane Let p be a plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$.

1. The Scalar equation of p is given by
   $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
2. Let $P = (x, y, z)$ be an arbitrary point. The vector equation of p is given by
   $$n \cdot \vec{P_0P} = 0.$$